Dimensional reduction of quantum fields on a brane

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Abstract

If we restrict a quantum field defined on a regular D dimensional curved manifold to a d dimensional submanifold then the resulting field will still have the singularity of the original D dimensional model. We show that a singular background metric can force the restricted field to behave as a d dimensional quantum field.

1 Introduction

Quantum fields are defined by their correlation functions. The Lagrangian serves as a heuristic tool for a construction of quantum fields. A reduction of the number of coordinates in the D dimensional Lagrangian does not mean that if we had a complete D-dimensional quantum field theory then we could reduce it in any way to a model resembling a quantum field theory in $d < D$ dimensions. We can see this problem already at the level of a massless free field ϕ. The vacuum correlation function of $\phi(\mathbf{x}(1))$ and $\phi(\mathbf{x}(2))$ is $|\mathbf{x}(1) - \mathbf{x}(2)|^{-D+2}$. If we restrict the field to the hypersurface $x_D = 0$ setting in all correlation functions $x_D(j) = 0$ then we obtain a quantum field with a continuous mass spectrum in $d = D-1$ dimensions but this will not be the canonical free field in d dimensions whose two-point function behaves as $|\mathbf{x}(1) - \mathbf{x}(2)|^{-D+3}$ at short distances. Nevertheless, it is an attractive idea that the Universe once had more dimensions and subsequently through a dynamical process shrank to a lower dimensional hypersurface. The dynamics could have the form of a gravitational collapse (say a ball collapsing to a disk). At the level of field correlation functions this would mean that we have initially scalar, electromagnetic and gravitational fields in D-dimensions with their standard canonical singularities which subsequently evolve into fields with $d < D$ dimensional singularity. We show that such a reduction of dimensions is possible when the metric becomes singular. A similar mechanism is suggested in the brane scheme of refs.[1][2]. In ref.[1]

the authors derive the Green's function in $D = 5$ dimensional space-time which on the $d = D - 1 = 4$ submanifold has the singularity of the fourdimensional Green's function. Their model encounters some difficulties when generalized to arbitrary D and d [3]. Some other brane-type models of quantum fields are discussed in refs.[4] [5][6][7]. In this letter we discuss a general metric which has power-law singularity. In general relativity such metrics could describe collapse phenomena [8]. We can obtain metrics with power-law singularities as solutions to higher dimensional supergravity theories [9]. These solutions describe m-branes or intersecting m-branes in an $m + n$ dimensional space-time [10]).

2 A quantum field on a D-1 dimensional hypersurface

We consider a submanifold $\mathcal{M}_{D-1}$ of a Riemannian manifold $\mathcal{M}_D$ whose metric becomes singular near $\mathcal{M}_{D-1}$. The metric on $\mathcal{M}_D$ close to $\mathcal{M}_{D-1}$ (in local coordinates) is described by a "warp factor" $a(x_D)$ which becomes singular either when $x_D \to 0$ or $|x_D| \to \infty$

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = dx_D^2 + a(x_D)^2 (dx_1^2 + ... + dx_{D-1}^2) \tag{1}$$

The Green's function of the minimally coupled scalar field is a solution of the equation

$$\mathcal{A}G = g^{-\frac{1}{2}} \delta \tag{2}$$

where $\mathcal{A}$ is the Laplace-Beltrami operator

$$\mathcal{A} = g^{-\frac{1}{2}} \partial_\mu (g^{\mu\nu} g^{\frac{1}{2}} \partial_\nu) \tag{3}$$

In the metric (1) eq.(2) reads

$$(\partial_D a^{D-1} \partial_D + a^{D-3} \triangle) G = \delta(x_D - x'_D) \delta(\mathbf{x} - \mathbf{x}') \tag{4}$$

where $d = D - 1$, $\mathbf{x} = (x_1, ..., x_d)$ and $\triangle$ is the d-dimensional Laplacian. This equation is simplified if we introduce the coordinate

$$\eta = \int a^{-d} dx_D \tag{5}$$

Then

$$(\partial_\eta^2 + a^{2d-2} \triangle) G = \delta(\eta - \eta') \delta(\mathbf{x} - \mathbf{x}') \tag{6}$$

In the paper of Dvali et al [1]$D = 4$ and $a^4(x_D(\eta)) \to \delta(\eta)$.

We discuss in detail the case

$$a(x_D) = |x_D|^\alpha \tag{7}$$

Then

$$(\partial_D|x_D|^{\alpha d}\partial_D+|x_D|^{\alpha(d-2)}\triangle)G=\delta(x_D-x'_D)\delta(\mathbf{x}-\mathbf{x}') \tag{8}$$

We define

$$\eta=|1-\alpha d|^{-1}x_D|x_D|^{-\alpha d} \tag{9}$$

then eq.(8) takes the form

$$(\partial^2_\eta+\kappa|\eta|^{2\nu}\triangle)G=\delta(\eta-\eta')\delta(\mathbf{x}-\mathbf{x}') \tag{10}$$

or in terms of the Fourier transform $\tilde{G}$ in $\mathbf{x}$

$$(\partial^2_\eta-\mathbf{p}^2V(\eta))\tilde{G}=\delta(\eta-\eta') \tag{11}$$

where

$$V(\eta)=\kappa|\eta|^{2\nu} \tag{12}$$

with

$$\nu=\alpha(d-1)(1-\alpha d)^{-1} \tag{13}$$

and

$$\kappa=|1-\alpha d|^{-2\nu}$$

Eq.(11) can be solved by means of the Feynman-Kac integral applying the proper time method

$$\begin{aligned}&G(\eta,\mathbf{x};\eta',\mathbf{x}')=\tfrac{1}{2}(2\pi)^{-d}\int_0^\infty d\tau\int d\mathbf{p}\exp\left(i\mathbf{p}\left(\mathbf{x}'-\mathbf{x}\right)\right)\\&E[\delta\left(\eta'-\eta-b\left(\tau\right)\right)\exp\left(-\tfrac{1}{2}\mathbf{p}^2\int_0^\tau V\left(\eta+b\left(s\right)\right)ds\right)]\end{aligned} \tag{14}$$

Here, $b(s)$ is the Brownian motion [11] defined as the Gaussian process with the covariance

$$E[b(s)b(t)]=min(s,t) \tag{15}$$

$E[..]$ denotes an average over the paths of the Brownian motion.

The dimensional reduction is imposed by setting $\eta=\eta'=0$. Next, we use the equivalence $b(s)=\sqrt{\tau}b(\frac{s}{\tau})$ which follows from eq.(15). Then, using the scaling invariance of the potential V (i.e.,$V(\lambda\eta)=\lambda^{2\nu}V(\eta)$) we have

$$\begin{aligned}&G(0,\mathbf{x};0,\mathbf{x}')=\tfrac{1}{2}(2\pi)^{-d}\int_0^\infty d\tau\int d\mathbf{p}\exp\left(i\mathbf{p}\left(\mathbf{x}'-\mathbf{x}\right)\right)\\&E[\delta\left(\sqrt{\tau}b\left(1\right)\right)\exp\left(-\tfrac{1}{2}\tau^{1+\nu}\mathbf{p}^2\int_0^1 V\left(b\left(s\right)\right)ds\right)]\end{aligned} \tag{16}$$

Changing the variables

$$\mathbf{p}=\tau^{-\frac{1}{2}-\frac{\nu}{2}}\mathbf{k}$$

and

$$\tau=r|\mathbf{x}-\mathbf{x}'|^{\frac{2}{1+\nu}}$$

we obtain

$$G(0,\mathbf{x};0,\mathbf{x}')=C|\mathbf{x}'-\mathbf{x}|^{-d+\frac{1}{1+\nu}} \tag{17}$$

with a certain constant C. If $0 > \nu > -1$ then the singularity of the Green's function is weaker than the one for the D-dimensional free field. The Green's function is equal to the Green's function of the $d = D - 1$ dimensional free field if $\nu = -\frac{1}{2}$ what corresponds to $\alpha = \frac{1}{2-d}$. The potential with $2\nu = -1$ has the same scaling dimension as $V = \delta(\eta)$ applied by Dvali et al [1]. The Hamiltonian with the potential $V(\eta) = |\eta|^{-1}$ and the path integral (16) require a careful definition if $2\nu \leq -1$ but at least till $2\nu \geq -2$ such a definition (through a regularization and a subsequent limiting procedure) is possible [12]. Eq.(11) with the δ-potential ("δ-brane") also involves a particular regularization and its subsequent removal [13]. Let us consider a solution of this problem by means of the proper time method. The heat kernel K^δ is known exactly for the δ-potential [14]. Hence,

$$\begin{aligned} &G(\eta, \mathbf{x}; \eta', \mathbf{x}') = \tfrac{1}{2}(2\pi)^{-4} \int_0^\infty d\tau \int d\mathbf{p} \exp\left(i\mathbf{p}\left(\mathbf{x}' - \mathbf{x}\right)\right) K^\delta(\eta, \eta', \tau) \\ &= \tfrac{1}{2}(2\pi)^{-4} \int_0^\infty d\tau \int d\mathbf{p} \exp\left(i\mathbf{p}\left(\mathbf{x}' - \mathbf{x}\right)\right) \\ &\Big(K_0(\eta - \eta', \tau) - 2\mathbf{p}^2 \int_0^\infty du \exp(-2\mathbf{p}^2 u) K_0(|\eta| + |\eta'| + u, \tau)\Big) \end{aligned} \tag{18}$$

where

$$K_0(\eta, \tau) = (2\pi\tau)^{-\frac{1}{2}} \exp(-\frac{1}{2\tau}\eta^2) \tag{19}$$

is the heat kernel for the Brownian motion.

When $\eta = \eta' = 0$ the τ-integral of the first term on the r.h.s. of eq.(18) (the one independent of $\mathbf{p}$) is infinite (and proportional to $\delta(\mathbf{x} - \mathbf{x}')$) whereas the second integral gives the formula(17) with $\nu = -\frac{1}{2}$.

Eq.(18) could have been derived as a limiting case of eqs. (6) and(16) when $a(x_D)^{2d-2} \to \delta(\eta)$. On the Lagrangian level we have

$$\int dx_D d\mathbf{x} \sqrt{g} g^{DD} \partial_D \phi \partial_D \phi = \int d\eta d\mathbf{x} \partial_\eta \phi \partial_\eta \phi \tag{20}$$

and

$$\int dx_D d\mathbf{x} \sqrt{g} g^{jk} \partial_j \phi \partial_k \phi = \int d\eta d\mathbf{x} a^{2d-2} \partial_j \phi \partial_j \phi \to \int d\eta d\mathbf{x} \delta(\eta) \partial_j \phi \partial_j \phi \tag{21}$$

Hence, we recover the Lagrangian of Dvali et al [1].

3 A generalization to surfaces of arbitrary dimensions

Let us consider on a $D = m + n$ dimensional manifold a metric (in local coordinates) which close to the n-dimensional surface takes the form

$$ds^2 = |\mathbf{y}|^{2\beta} d\mathbf{y}^2 + |\mathbf{y}|^{2\alpha} d\mathbf{x}^2 \tag{22}$$

where $\mathbf{y} \in R^m$ and $\mathbf{x} \in R^n$. Eq.(2) for the Green's function of the Laplace-Beltrami operator reads

$$\Big(\frac{\partial}{\partial y^i}|\mathbf{y}|^{\beta(m-2)+\alpha n}\frac{\partial}{\partial y^i} + |\mathbf{y}|^{\beta m+\alpha(n-2)}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^j}\Big)G_E = \delta \tag{23}$$

We discuss here only a simplified form of eq.(23) which appears when

$$\beta(m-2) + \alpha n = 0 \tag{24}$$

In such a case eq.(23) reads

$$\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^i} + |\mathbf{y}|^{2\beta-2\alpha}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^j}\Big)G_E = \delta \tag{25}$$

or taking the Fourier transform in $\mathbf{x}$

$$\Big(\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_i} - \mathbf{p}^2 V(\mathbf{y})\Big)\tilde{G}_E = \delta(\mathbf{y}) \tag{26}$$

We obtain again an equation for the Green's function of the Schrödinger operator with the potential

$$V(\mathbf{y}) = |\mathbf{y}|^{2\beta-2\alpha} \tag{27}$$

and the coupling constant $\mathbf{p}^2$. We solve eq.(26) by means of the proper time method

$$\begin{aligned} &G(\mathbf{y},\mathbf{x};\mathbf{y}',\mathbf{x}') = \tfrac{1}{2}(2\pi)^{-n}\int_0^\infty d\tau \int d\mathbf{p}\exp\left(i\mathbf{p}\left(\mathbf{x}'-\mathbf{x}\right)\right) \\ &E[\delta\left(\mathbf{y}'-\mathbf{y}-\mathbf{b}\left(\tau\right)\right)\exp\left(-\tfrac{1}{2}\mathbf{p}^2\int_0^\tau V\left(\mathbf{y}+\mathbf{b}\left(s\right)\right)ds\right)] \end{aligned} \tag{28}$$

where $\mathbf{b}$ is the m-dimensional Brownian motion.

On the brane $\mathbf{y} = \mathbf{y}' = \mathbf{0}$. In such a case using $\mathbf{b}(s) = \sqrt{\tau}\mathbf{b}(\frac{s}{\tau})$ we have

$$\int_0^\tau ds V(\mathbf{b}(s)) = \tau^{1+\beta-\alpha}\int_0^1 V(\mathbf{b}(s))ds \tag{29}$$

Hence, if we change variables

$$\mathbf{p} = \mathbf{k}\tau^{-\frac{1}{2}(1+\beta-\alpha)}$$

then

$$\begin{aligned} &G(\mathbf{0},\mathbf{x};\mathbf{0},\mathbf{x}')G(\mathbf{0},\mathbf{x};\mathbf{0},\mathbf{x}') \\ &= \tfrac{1}{2}(2\pi)^{-n}\int_0^\infty d\tau\sqrt{\tau}^{n(\alpha-1-\beta)-m}\int d\mathbf{k}\exp\left(i\mathbf{k}\sqrt{\tau}^{\alpha-1-\beta}\left(\mathbf{x}'-\mathbf{x}\right)\right) \\ &E[\delta\left(\mathbf{b}\left(1\right)\right)\exp\left(-\tfrac{1}{2}\mathbf{k}^2\int_0^1 V\left(\mathbf{b}\left(s\right)\right)ds\right)] = C|\mathbf{x}-\mathbf{x}'|^{-n+\rho} \end{aligned} \tag{30}$$

with a certain constant C and

$$\rho = (2-m)(1-\alpha+\beta)^{-1}$$

For canonical quantum fields in n dimensions we should have $\rho = 2$. This happens if (in addition to eq.(24))

$$\alpha - \beta = \frac{m}{2} \tag{31}$$

In such a case the potential is

$$V(\mathbf{y}) = |\mathbf{y}|^{-m} \tag{32}$$

The potential (32) scales in the same way as the δ-function in m-dimensions. This is a singular potential. However, its regularization $V_\epsilon(\mathbf{y}) = |\mathbf{y}|^{-m-\epsilon}$ for any $\epsilon > 0$ gives a self-adjoint Hamiltonian with the well-defined path integral. As ϵ can be arbitrarily small the Newton potential on the brane would be indistinguishable from r^{-1} if the brane is $n-1=3$ dimensional. We could again consider the limit $V(\mathbf{y}) \rightarrow \delta(\mathbf{y})$ in order to derive the model of Dvali et al [3]. In contradistinction to the case $m=1$ the models in $m>1$ dimensions are more complicated. For $m=2$ and $m=3$ the relation of the coupling constant $\mathbf{p}^2$ in eq.(26) to the parameters appearing in the heat kernel K^δ is not so explicit [15]. For $m>3$ the δ-potential cannot be defined at all [16] [13].

We have discussed only scale invariant metrics . If the metric is not scale invariant but its asymptotic behaviour for $\mathbf{y} \rightarrow 0$ is of the form (22) then our results hold true when $-1 \leq \nu \leq 0$ and when applied to the short distance behaviour $|\mathbf{x}-\mathbf{x}'| \rightarrow 0$ of $G(\mathbf{0},\mathbf{x};\mathbf{0},\mathbf{x}')$. If the asymptotic behaviour of the metric for $|\mathbf{y}| \rightarrow \infty$ is of the form (22) then our results apply if $\beta \geq \alpha$ to the behaviour of the Green's functions $G(\mathbf{0},\mathbf{x};\mathbf{0},\mathbf{x}')$ for large $|\mathbf{x}-\mathbf{x}'|$. In such a case $\rho < 2$ in eq.(30), hence $G(\mathbf{0},\mathbf{x};\mathbf{0},\mathbf{x}')$ and the gravitational potential decay to zero faster than in the Newton theory (in the n dimensions). Depending on the asymptotic behaviour of the metric tensor $g(\mathbf{x},\mathbf{y})$ we obtain models which lead to a modification of the Newton law either at small or at large distances (some brane models modifying the classical gravity at small or large distances are discussed in [4][7]).

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